On Algebras and Logics based on Fuzzy Equality

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Abstract:

Mathematical logic has been for many years developed on the basis of implication as the main connective. In eighties, another direction has been initiated which is called equational logic. The emphasis is put on the style of proofs in which substitution of equals for equals, instead of Modus Ponens is applied. Thus, equality (or equivalence) takes an important role. This brings an idea whether also fuzzy logic could be developed on the basis of fuzzy equality as the principal connective (in truth values, fuzzy equality becomes fuzzy equivalence). For this purpose, a special algebra called EQ-algebra has been introduced, in which fuzzy equality is the basic operation and implication is derived from it. In general, this is a semilattice extended by two additional operations: a binary monoidal operation of multiplication and a binary operation of fuzzy equality. The implication is derived from the fuzzy equality and so, it is not tied with the multiplication as in residuated lattices. EQ-algebras are more general than the latter, i.e., each residuated lattice is an EQ-algebra but not vice-versa.

There are several special kinds of EQ-algebra with interesting properties. Surprisingly, only some of them are suitable candidates for the development of logic. We call the logic based on EQ-algebras EQ-logic. Its basic connectives are equivalence, conjunction and fusion while implication is a derived operation. The basic EQ-logic lays even deeper than the MTL-fuzzy logic which is obtained after adding few more axioms to the former.

A special connective having important role in EQ-logics is that of Baaz delta (on linearly ordered algebras, it keeps 1 and sends all other values to 0). Up to now, several kinds of propositional EQ-logics have been developed, either with or without Baaz delta. Moreover, a higher-order fuzzy logic, namely the EQ-fuzzy type theory has been developed, too. Its syntax is analogous to that of the classical type theory but it has more connectives and axioms. In classical as well as in fuzzy type theories, though, (fuzzy) equality is fundamental. Generalized completeness theorem holds for all kinds of EQ-logics.